

Discrete symmetries in the three-Higgs-doublet model

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N -Higgs-doublet models (NHDM) are among the most popular examples of electroweak symmetry breaking mechanisms beyond the Standard Model. Discrete symmetries imposed on the NHDM scalar potential play a pivotal role in shaping the phenomenology of the model, and various symmetry groups have been studied so far. However, in spite of all efforts, the classification of finite Higgs-family symmetry groups realizable in NHDM for any $N > 2$ is still missing. Here, we solve this problem for the three-Higgs-doublet model. Using recently found realizable abelian groups and applying Burnside's theorem and other group-theoretic tools, we find the full list of finite symmetry groups of Higgs-family transformations which are realizable in the scalar sector of 3HDM.

Introduction. — The nature of the electroweak symmetry breaking remains one of the hottest issues in high-energy physics. It is believed to be mediated by the Higgs mechanism, and many different variants of it have been proposed so far [1]. One conceptually simple and phenomenologically attractive class of models involves several Higgs doublets with identical quantum numbers. The scalar potential in these N -Higgs-doublet models (NHDM) is often assumed to be symmetric under a group of unitary (Higgs-family) or anti-unitary (generalized CP) transformations acting in the space of doublets. These symmetries play a pivotal role in the phenomenology of the model, both in the scalar and in the fermionic sectors, and they often bear interesting astrophysical consequences. Given the importance of symmetries for the NHDM phenomenology, it is natural to ask: which symmetry groups can be implemented in the scalar sector of NHDM for a given N ?

In the two-Higgs-doublet model (2HDM), this question has been answered several years ago, [2], see also [3] for a review. Focusing on discrete symmetries, the only realizable group of unitary symmetries is \mathbb{Z}_2 . If anti-unitary transformations are included, then $(\mathbb{Z}_2)^2$ and $(\mathbb{Z}_2)^3$ are also realizable. For each group, the corresponding potential was written and phenomenological consequences were studied in detail (for example, an investigation of the $(\mathbb{Z}_2)^3$ -symmetric 2HDM can be found see [4]).

With more than two doublets, the problem remains open. Variants of NHDM based on several finite groups have been studied, with an emphasis on the A_4 group, [5, 6], and several attempts have been made to classify at least some symmetries in NHDM, [7]. In particular, a classification of all realizable abelian symmetry groups in NHDM for any N was recently given in [8]. However, the full list of non-abelian finite groups which can be symmetry groups in NHDM scalar sector is not yet known. We stress that this task is different from just classifying all finite subgroups of $SU(3)$, [9], because invariance of the Higgs potential places strong and non-trivial restrictions on possible symmetry groups.

In this paper we solve this problem for the three-Higgs-doublet model (3HDM). Starting from abelian groups and applying several results from the finite group theory, we find the complete list of discrete symmetry groups of Higgs-family transformations realizable in 3HDM. Extension of our method to groups which include anti-unitary

transformations will be given elsewhere.

Structure of finite symmetry groups in 3HDM. — The most general renormalizable gauge-invariant scalar potential of 3HDM can be written as

$$V = Y_{ij}(\phi_i^\dagger \phi_j) + Z_{ijkl}(\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l), \quad (1)$$

where all indices run from 1 to 3. We are interested in unitary transformations mixing doublets ϕ_i that leave this potential invariant for some Y_{ij} and Z_{ijkl} . A priori, these transformations belong to the group $U(3)$. Multiplying the three doublets by a common phase factor, which trivially leaves the potential invariant, is already taken into account in the gauge group $U(1)_Y$. Therefore, we focus on additional transformations not reducible to overall phase rotations, which form the group $PSU(3) = SU(3)/\mathbb{Z}_3$, where \mathbb{Z}_3 is the center of $SU(3)$. Our task is therefore to find finite subgroups of $PSU(3)$ which can be the symmetry groups of the potential (1) for some choices of coefficients. We stress that we search for *realizable* symmetry groups, that is, for groups $G \subset PSU(3)$ such that there exists a G -symmetric potential which is not invariant under a larger symmetry group $G' \supset G$, see a fuller discussion in [8].

Abelian realizable symmetry groups for NHDM were characterized in [8]. For our task of classifying finite realizable symmetry groups in 3HDM, the following abelian groups must be considered:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3. \quad (2)$$

The first four are the only realizable finite subgroups of maximal tori in $PSU(3)$. The last group, $\mathbb{Z}_3 \times \mathbb{Z}_3$, is on its own a maximal abelian subgroup of $PSU(3)$, but it is not realizable because a $\mathbb{Z}_3 \times \mathbb{Z}_3$ -symmetric potential is automatically symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, see explicit expressions below. However, it still can appear as an abelian subgroup of a finite non-abelian realizable group, therefore it must be included into consideration. Trying to impose any other abelian Higgs-family symmetry group on the 3HDM potential unavoidably makes it symmetric under a continuous group.

Let us denote by $G \subset PSU(3)$ a finite (non-abelian) symmetry group in 3HDM. We shall now apply some results from the finite group theory to prove that G cannot be too large, and more specifically, we shall describe the generic structure of G .

All abelian subgroups of G must be from the list (2). By Chauchy's theorem, if p is a prime divisor of the order of the group, $|G|$, then G contains a subgroup \mathbb{Z}_p . Thus, the order of the group can have only two prime divisors: $|G| = 2^a 3^b$. Then according to the Burnside $p^a q^b$ -theorem, the group G is *solvable*. Solvability implies that G contains a normal abelian subgroup, which belongs, of course, to the list (2). This is our first key group-theoretic step.

Suppose A is the normal abelian subgroup of G , $A \triangleleft G$. Obviously, $A \subseteq C_G(A)$, the centralizer of A in G (all elements $g \in G$ which commute with all $a \in A$). As we prove in Appendix, we can assume that A is self-centralizing: $A = C_G(A)$. This means that elements $g \in G$, $g \notin A$, cannot commute with *all* elements of A . Therefore, they induce automorphisms (i.e. structure-preserving permutations) on A : $g^{-1}ag \in A$ for any $a \in A$, and these automorphisms are non-trivial. Even more, if g_1 and g_2 induce the same automorphism on A , $g_1^{-1}ag_1 = g_2^{-1}ag_2$ for all $a \in A$, then g_1 and g_2 belong to the same coset of A in G : $g_2 = g_1 a'$. Therefore, the homomorphism $f : G/A \rightarrow \text{Aut}(A)$, where $\text{Aut}(A)$ is the group of automorphisms on A , is *injective*. We conclude that

$$G/A = K, \quad K \subseteq \text{Aut}(A). \quad (3)$$

This is our second key group-theoretic step. It proves that G cannot be too large, and it also shows that G can be constructed as an extension of A by a subgroup of $\text{Aut}(A)$: $G = A \cdot K$.

We now check all the candidates for A from the list (2) and see which extension can work in 3HDM. We use the explicit realization of each of the groups A , [8], and search for additional transformations from $PSU(3)$ with the desired multiplication properties.

Extending \mathbb{Z}_2 . — If $A = \mathbb{Z}_2$, then $\text{Aut}(\mathbb{Z}_2) = \{1\}$, so that $G = \mathbb{Z}_2$. This case was already considered in [8].

Extending \mathbb{Z}_3 . — If $A = \mathbb{Z}_3$, then $\text{Aut}(\mathbb{Z}_3) = \mathbb{Z}_2$. The only non-trivial case to be considered is $G/A = \mathbb{Z}_2$, so that G is the dihedral group representing the symmetries of an equilateral triangle $G = D_6 = S_3$. If \mathbb{Z}_3 group is generated by the phase rotations $a = \text{diag}(\omega, \omega^2, 1)$ with $\omega = 2\pi/3$, then the transformation b generating \mathbb{Z}_2 and satisfying $b^{-1}ab = a^2$ must be of the form

$$b = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (4)$$

with arbitrary δ . The choice of the mixing pair of doublets (ϕ_1 and ϕ_2 in this case) is also arbitrary, so other b 's with different pairs of mixing doublets are also allowed. The fact that b is not uniquely defined means that there is a whole family of D_6 groups parametrized by the value of δ even if we start with the fixed group $A = \mathbb{Z}_3$.

The generic \mathbb{Z}_3 -symmetric potential contains the part invariant under any phase rotation

$$V_0 = - \sum_i m_i^2 (\phi_i^\dagger \phi_i) + \sum_{i,j} \lambda_{ij} (\phi_i^\dagger \phi_i) (\phi_j^\dagger \phi_j) + \sum_{i \neq j} \lambda'_{ij} (\phi_i^\dagger \phi_j) (\phi_j^\dagger \phi_i), \quad (5)$$

and the following additional terms

$$V_{\mathbb{Z}_3} = \lambda_1 (\phi_2^\dagger \phi_1) (\phi_3^\dagger \phi_1) + \lambda_2 (\phi_1^\dagger \phi_2) (\phi_3^\dagger \phi_2) + \lambda_3 (\phi_1^\dagger \phi_3) (\phi_2^\dagger \phi_3) + h.c. \quad (6)$$

with complex $\lambda_1, \lambda_2, \lambda_3$. If the parameters of V_0 satisfy

$$m_{11}^2 = m_{22}^2, \quad \lambda_{11} = \lambda_{22}, \quad \lambda_{13} = \lambda_{23}, \quad \lambda'_{13} = \lambda'_{23}, \quad (7)$$

and if, in addition, moduli of two among the three coefficients $\lambda_1, \lambda_2, \lambda_3$ coincide, for example $|\lambda_1| = |\lambda_2|$, then the potential $V_0 + V_{\mathbb{Z}_3}$ becomes symmetric under one particular D_6 group constructed with b in (4) with the value of $\delta = (\arg \lambda_2 - \arg \lambda_1 + \pi)/3$.

This construction allows us to write down an example of the D_6 potential. In order to prove that D_6 is indeed a realizable group, we need to show that the resulting potential is not symmetric under any other Higgs-family transformation. This is proved by the mere observation that all other possible groups to be discussed below which could contain D_6 lead to *stronger* restrictions on the potential than (7) and $|\lambda_1| = |\lambda_2|$. Therefore, not satisfying those stronger restrictions will yield a potential symmetric only under D_6 . Finally, one can also show that the potential we obtained does not have any continuous symmetry. The same logic applies to other realizable groups below.

Extending \mathbb{Z}_4 . — If $A = \mathbb{Z}_4$ (generated by a), then $\text{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$, so that $G = \mathbb{Z}_4 \cdot \mathbb{Z}_2$. The two non-abelian possibilities for G are the dihedral group D_8 representing symmetries of the square, and the quaternion group Q_8 . In both cases $b^{-1}ab = a^3$, with the only difference that $b^2 = 1$ for D_8 while $b^2 = a^2$ for Q_8 . Representing a by $\text{diag}(i, -i, 1)$, we find

$$b(D_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad b(Q_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ -e^{-i\delta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, in each case we obtain a family of b 's parametrized by phase δ . The \mathbb{Z}_4 -symmetric potential is $V_0 + V_{\mathbb{Z}_4}$ where

$$V_{\mathbb{Z}_4} = \lambda_1 (\phi_3^\dagger \phi_1) (\phi_3^\dagger \phi_2) + \lambda_2 (\phi_1^\dagger \phi_2)^2 + h.c. \quad (8)$$

An explicit analysis shows that to make it D_8 -invariant, we only need to satisfy conditions (7). Then, the potential is symmetric under $b(D_8)$ with the phase $\delta = \arg \lambda_2/2$. Since any larger group that could possibly contain D_8 leads to stronger restrictions on the potential, we conclude that D_8 is realizable in 3HDM.

Now, if instead of D_8 we try to make the potential symmetric under Q_8 , we unavoidably need to set $\lambda_1 = 0$. Removing one term from (8) immediately makes it symmetric under a *continuous* group of phase rotations, [8]. Therefore, Q_8 is not realizable in 3HDM.

Extending $\mathbb{Z}_2 \times \mathbb{Z}_2$. — If $A = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = GL_2(2) = S_3$. $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be realized as the group of independent sign flips of the three doublets with generators $a_1 = \text{diag}(1, -1, -1)$ and $a_2 = \text{diag}(-1, 1, -1)$. The potential symmetric under this group contains V_0 and additional terms

$$V_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \tilde{\lambda}_{12} (\phi_1^\dagger \phi_2)^2 + \tilde{\lambda}_{23} (\phi_2^\dagger \phi_3)^2 + \tilde{\lambda}_{31} (\phi_3^\dagger \phi_1)^2 + h.c. \quad (9)$$

The coefficients $\tilde{\lambda}_{ij}$ can be complex; we denote their phases as ψ_{ij} .

There are three possibilities to extend A : by \mathbb{Z}_2 , by \mathbb{Z}_3 , and by S_3 . The first extension, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \cdot \mathbb{Z}_2$, leads to D_8 , which was already constructed above.

The extension by \mathbb{Z}_3 is necessarily split, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, leading to the group $T \simeq A_4$, the symmetry group of a tetrahedron. To construct it, we need to find b acting on $\{a_1, a_2, a_1 a_2\}$ by cyclic permutations. Fixing the direction of permutations by $b^{-1} a_1 b = a_2$, we find that b must be of the form

$$b = \begin{pmatrix} 0 & e^{i\delta_1} & 0 \\ 0 & 0 & e^{i\delta_2} \\ e^{-i(\delta_1+\delta_2)} & 0 & 0 \end{pmatrix}, \quad (10)$$

with arbitrary δ_1, δ_2 . It then follows that if coefficients in (9) satisfy

$$|\tilde{\lambda}_{12}| = |\tilde{\lambda}_{23}| = |\tilde{\lambda}_{31}|, \quad (11)$$

then $V_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is symmetric under a particular b with

$$\delta_1 = \frac{2\psi_{12} - \psi_{31} - \psi_{23}}{6}, \quad \delta_2 = \frac{2\psi_{23} - \psi_{31} - \psi_{12}}{6}.$$

By rephasing, one can bring (9) to the following form

$$V_T = \tilde{\lambda} \left[(\phi_1^\dagger \phi_2)^2 + (\phi_2^\dagger \phi_3)^2 + (\phi_3^\dagger \phi_1)^2 \right] + h.c. \quad (12)$$

with complex $\tilde{\lambda}$. In addition, the symmetry under b places stronger conditions on the parameters of V_0 , and the most general V_0 satisfying them is now

$$V_0 = -m^2 \sum_i (\phi_i^\dagger \phi_i) + \lambda \left[\sum_i (\phi_i^\dagger \phi_i) \right]^2 + \sum_{i \neq j} \left[\lambda' (\phi_i^\dagger \phi_i) (\phi_j^\dagger \phi_j) + \lambda'' |\phi_i^\dagger \phi_j|^2 \right]. \quad (13)$$

with real $m^2, \lambda_0, \lambda_1, \lambda_2$ and complex λ_3 , all values being generic. This potential is, however, symmetric under a larger group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, which is generated by $\bar{a}, \bar{b}, \bar{c}$ with the following relations

$$\bar{a}^3 = \bar{b}^3 = 1, \quad \bar{c}^2 = 1, \quad [\bar{a}, \bar{b}] = 1, \quad \bar{c} \bar{a} \bar{c}^{-1} = \bar{a}^2, \quad \bar{c} \bar{b} \bar{c}^{-1} = \bar{b}^2.$$

In terms of the explicit transformation laws, \bar{c} is the coset $cZ(SU(3))$, with c being the exchange of any two doublets, so that $\langle \bar{a}, \bar{c} \rangle = S_3$ is the group of arbitrary permutations of the three doublets. Thus, if $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot K$, then a G -symmetric potential must be a restriction of (14), so that $K \supseteq \mathbb{Z}_2$.

The last extension, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \cdot S_3$, leads to the group $O = S_4$, the symmetry group of an octahedron and a cube. It includes T as a subgroup, therefore the most general O -symmetric potential is V_0 from (13) plus V_T from (12) with the additional condition that $\tilde{\lambda}$ is real.

Extending $\mathbb{Z}_3 \times \mathbb{Z}_3$. — Finally, if $A = \mathbb{Z}_3 \times \mathbb{Z}_3$, then $K \subset \text{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3) = GL_2(3)$, the general linear group of transformations of two-dimensional vector space over the finite field \mathbb{F}_3 , whose role is played by A . One can define an antisymmetric scalar product in this space and prove that K must include only transformations from $GL_2(3)$ that preserve this scalar product: $K \subseteq Sp_2(3) = SL_2(3)$.

The group $SL_2(3)$ has order 24 and contains elements of order 2, 3, 4, and 6. Elements of order 6 cannot be used for extension because they would generate the abelian subgroup \mathbb{Z}_6 , which is absent in (2). Besides, we will show below that K must always contain the subgroup \mathbb{Z}_2 . There are three kinds of subgroups of $K \subset SL_2(3)$ containing \mathbb{Z}_2 but not containing \mathbb{Z}_6 : \mathbb{Z}_2 , \mathbb{Z}_4 , and Q_8 . Since, as we argued, the quaternion group Q_8 is not realizable in 3HDM, K can only be \mathbb{Z}_2 or \mathbb{Z}_4 .

To show that $K \supseteq \mathbb{Z}_2$, consider first the group $P \subset SU(3)$ generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \omega = \exp\left(\frac{2\pi i}{3}\right).$$

Since $[a, b] = aba^{-1}b^{-1} \in Z(SU(3))$, the image of P under the canonical homomorphism $SU(3) \rightarrow PSU(3)$ becomes the desired abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$. The true generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are cosets $\bar{a} = aZ(SU(3))$ and $\bar{b} = bZ(SU(3))$ from $PSU(3)$. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ -invariant potential is

$$V = -m^2 \left[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3) \right] + \lambda_0 \left[(\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) + (\phi_3^\dagger \phi_3) \right]^2 + \frac{\lambda_1}{\sqrt{3}} \left[(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) - (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1) \right] + \lambda_2 \left[|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2 \right] + \lambda_3 \left[(\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + (\phi_2^\dagger \phi_3)(\phi_2^\dagger \phi_1) + (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2) \right] + h.c. \quad (14)$$

Turning now to the extension $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, we note that $SL_2(3)$ contains three distinct \mathbb{Z}_4 subgroups, which however intersect at the center of $SL_2(3)$. All three are conjugate inside $SL_2(3)$ and lead, up to isomorphism, to the same symmetry group. To give an example of a potential symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, we choose an element $d \in SL_2(3)$ of order 4 that generates the cyclic permutation of generators $\bar{a}, \bar{b}, \bar{a}^2, \bar{b}^2$. It can be

represented by the following $SU(3)$ transformation:

$$d = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad d^{-1} = d^*, \quad d^4 = 1. \quad (15)$$

Then by analyzing how the potential changes under d , we obtain the following criterion: (14) becomes symmetric under $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, if λ_3 is real and is equal to $(\sqrt{3}\lambda_1 - \lambda_2)/2$. One can also show that the resulting potential is not invariant under any continuous symmetry group.

Summary. — Because of important phenomenological role the symmetries play in multi-Higgs-doublet models, the task of classifying all symmetries in NHDM is of much interest. Here we solved this problem for 3HDM. Focusing on groups of unitary transformations and including the finite abelian groups found in [8], we obtain the following list of finite groups realizable as Higgs-family symmetry groups of the 3HDM scalar sector:

$$\begin{aligned} &\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \\ &D_6, \quad D_8, \quad T \simeq A_4, \quad O \simeq S_4, \\ &(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2, \quad (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4. \end{aligned} \quad (16)$$

This list is complete: trying to impose any other finite Higgs-family symmetry group on the 3HDM potential will unavoidably lead to a potential symmetric under a continuous group.

Applying methods described in [8], one can also obtain the list of realizable groups in 3HDM which include anti-unitary transformations. These results as well as a study of symmetry breaking patterns for each of these groups will be presented elsewhere.

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EXISTENCE OF NORMAL ABELIAN SELF-CENTRALIZING SUBGROUPS

In the main text we stated that the normal abelian subgroup $A \subset G$, whose existence follows from the Burnside's theorem, can be taken self-centralizing: $A = C_G(A)$. Here we prove this statement: namely, we show that even if $A \subset C_G(A)$, then there exists another abelian subgroup $A' \supset A$, which is normal and self-centralizing in G .

Suppose that $A \subset C_G(A)$. Then for every $b \in C_G(A)$ the group $A_b = \langle A, b \rangle$ is an abelian subgroup of G , which properly contains A . There are three possibilities compatible with the list (2): (i) $A = \mathbb{Z}_2$, and there exists $b \in C_G(A)$ such that $A_b = \mathbb{Z}_4$, (ii) $A = \mathbb{Z}_2$, A_b is necessarily $\mathbb{Z}_2 \times \mathbb{Z}_2$ and every non-trivial element from $C_G(A)$ has order 2, (iii) $A = \mathbb{Z}_3$, then $A_b = \mathbb{Z}_3 \times \mathbb{Z}_3$ and every non-trivial element of $C_G(A)$ has order 3.

In case (i) we have either $C_G(A) = \mathbb{Z}_4$ (then it is self-centralizing and normal in G and plays the role of the desired group A'), or $C_G(A) \supset \mathbb{Z}_4$, which automatically leads to a continuous symmetry in 3HDM and is, therefore, disregarded (we either get an abelian group absent from the list (2) or the quaternion group Q_8 , both possibilities leading to a continuous symmetry).

In case (ii) $C_G(A)$ is abelian (a group in which every nonzero element has order 2 is abelian) and is also normal in G (centralizer of a normal subgroup A is always normal as it is the kernel of the homomorphism $G \rightarrow \text{Aut}(A)$). Therefore, $A_b = C_G(A)$ is the desired normal abelian self-centralizing subgroup A' .

In case (iii) we note that every G -invariant potential is also $N_G(A_b)$ -invariant. $N_G(A_b)/A_b$ cannot have elements of order 3 (see the arguments in *Extending* $\mathbb{Z}_3 \times \mathbb{Z}_3$). Then, A_b is a Sylow 3-subgroup of $N_G(A_b)$. On the other hand, $A_b \subseteq C_G(A)$ and $C_G(A)$ is a 3-group, and so is its subgroup $N_{C_G(A)}(A_b)$. Therefore, $N_{C_G(A)}(A_b)/A_b$ cannot have elements of order 3 and, simultaneously, must be a 3-group. This means that $N_{C_G(A)}(A_b) = A_b$. Finally, since every proper subgroup of a p -group is properly contained in its normalizer, we conclude that $A_b = C_G(A) = C_G(A_b)$, and therefore, it is normal in G .